SAFE OPTIMAL CONTROL OF AUTONOMOUS VEHICLES WITH CONTROL BARRIER FUNCTIONS

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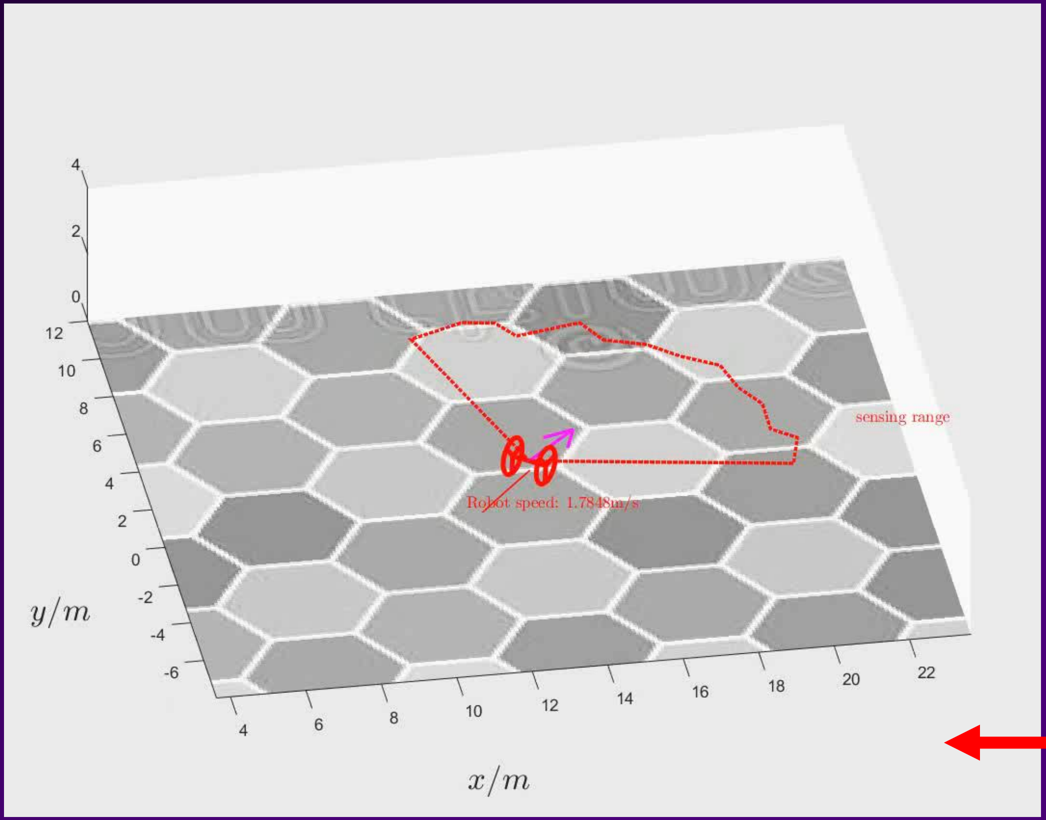
MOTIVATION: SAFE-CRITICAL TASKS



Reach **destination**…

…**safely**

(avoid obstacles with unknown locations)

…with **minimal** effort

(minimize energy)

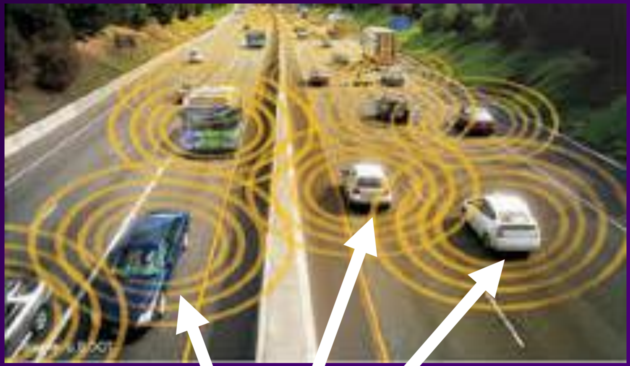
Actual execution using CBF-based optimal control

Design autonomous systems which can *guarantee*

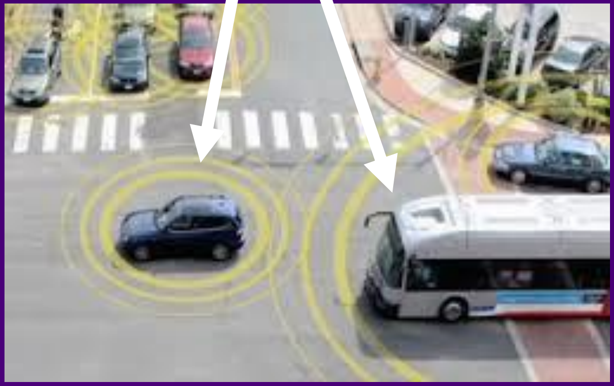
- **SAFE** and **OPTIMAL** control actions
- …throughout a task
- …with minimal or no supervision

MOTIVATION – THE INTERNET OF CARS

GAME-CHANGING OPPORTUNITY:
CONNECTED AUTONOMOUS VEHICLES (CAVs)



CAVs



FROM (SELFISH) “DRIVER OPTIMAL” TO (SOCIAL) “SYSTEM OPTIMAL” TRAFFIC CONTROL

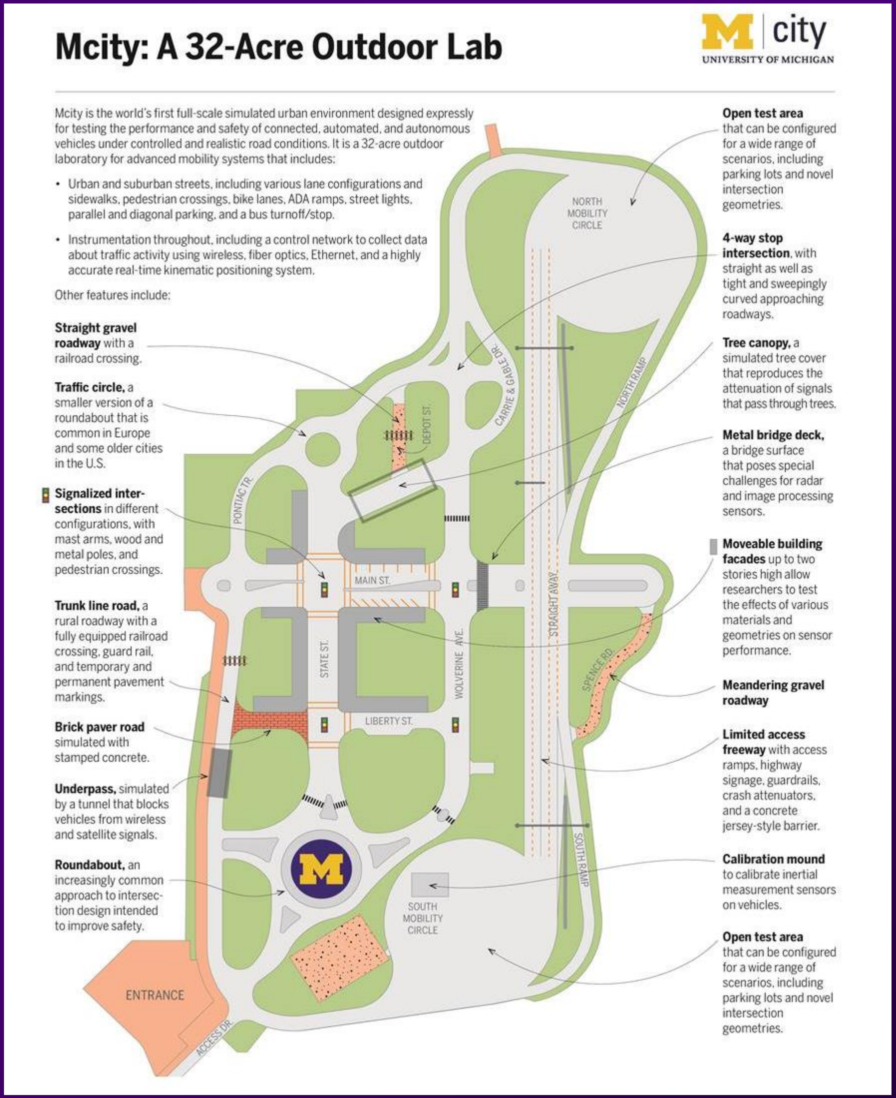
THE “INTERNET OF **CARS**”

NO TRAFFIC LIGHTS, NEVER STOP…

A TEST BED FOR CAVs

Mcity test bed, U. Michigan

2016 Audi A3 e-tron PHEV (ARPA-E project)



CONTROL ZONES

CONTROL ZONE (CZ):

Vehicles can cooperate to achieve desirable performance

Minimize Travel Time through CZ

+

Minimize Energy through CZ

+

Maximize Passenger Comfort

+

Guarantee Safety

FROM TRAJECTORY PLANNING TO EXECUTION: CONTROL BARRIER FUNCTIONS (CBFs)

Ames, Grizzle, Tabuada, CDC 2014; Xiao, Cassandras, Belta, *Automatica, 2021*

OPTIMAL CONTROL v REAL-TIME EXECUTION

OPTIMAL CONTROL

MODEL PREDICTIVE CONTROL

Optimality *BUT…*

- Hard to obtain solutions when constraints active
- Time consuming to compute
- Models are too simple
- Not robust to disturbances

- **Easy to solve *BUT…***

- Suboptimal
- Linearization errors
- Models still too simple
- Computationally expensive

OPTIMAL CONTROL v REAL-TIME EXECUTION

OPTIMAL CONTROL

CONTROL BARRIER FUNCTIONS

Optimality *BUT…*

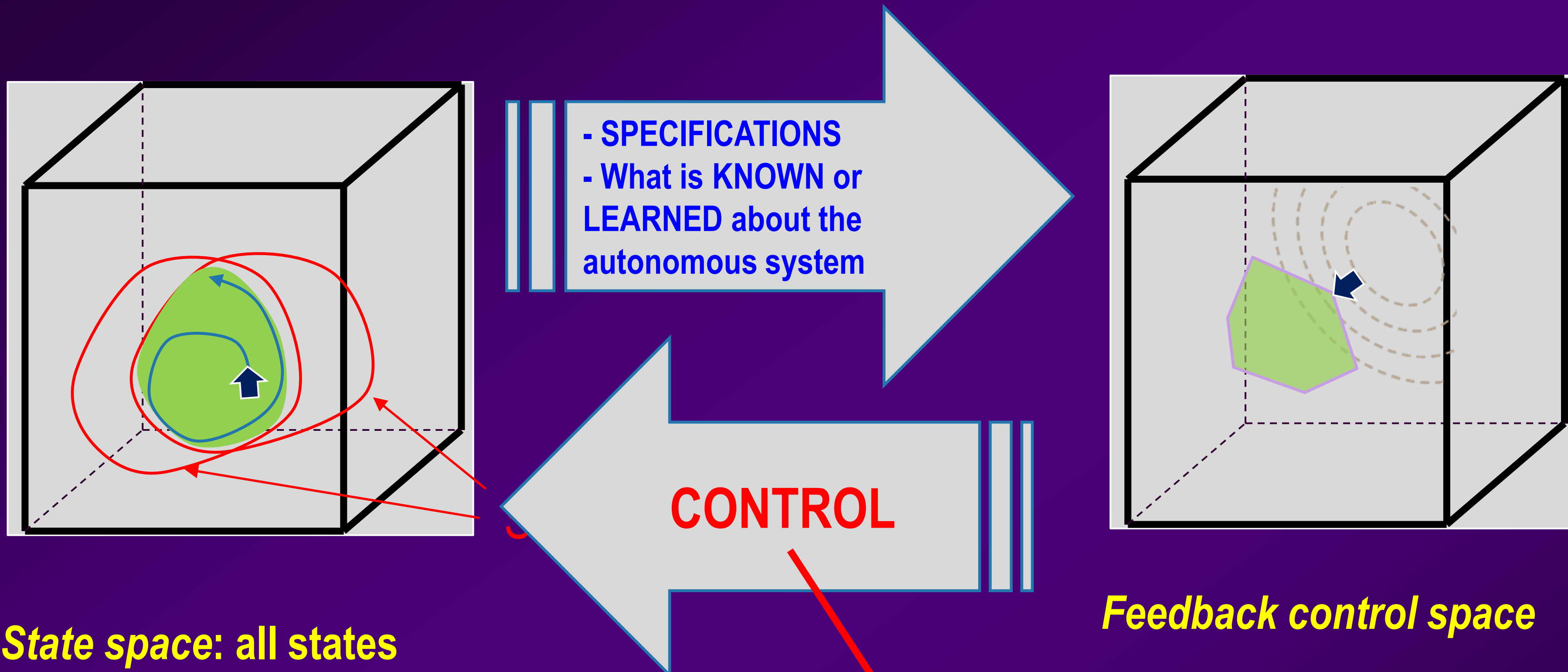
- Hard to obtain when constraints active
- Time consuming to compute
- Models are too simple
- Not robust to disturbances

- Easy to solve
- *Forward Invariance*
- Nonlinear Dynamics and Constraints
- Complex objectives
- Noise in dynamics

BUT…

- Suboptimal
- Possibly conservative, myopic

CONTROL BARRIER FUNCTIONS - OVERVIEW



State space: all states that autonomous system can access (e.g., positions, speeds,…)

FORWARD INVARIANCE:
Controller guarantees SAFETY over all future times

→ *Control Barrier Functions* (CBFs)

CONTROL BARRIER FUNCTIONS - OVERVIEW

- **System dynamics:** $\dot{x} = f(x) + g(x)u$
- **Control Barrier Functions:** (continuously differentiable *state* constraints) $b_k(x) \geq 0$
- **Map each CBF onto another constraint on the *control* input such that the satisfaction of this new constraint implies the satisfaction of the original constraint:**

IMPLIES

CBF constraints: $\boxed{L_f b_k(x(t)) + L_g b_k(x(t))u(t) + \gamma(b_k(x(t))) \geq 0}$

Lie derivatives along f and g

Class K function: strictly increasing, $\gamma(0) = 0, \quad \gamma(t) = \rightarrow \infty \text{ as } t \rightarrow \infty$

$$L_f V(x) \triangleq \frac{\partial V}{\partial x} f(x) \quad \text{and} \quad L_g V(x) \triangleq \frac{\partial V}{\partial x} g(x)$$

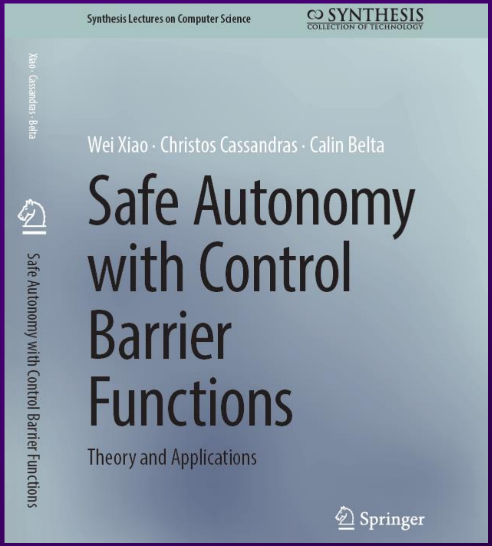
CONTROL BARRIER FUNCTIONS - OVERVIEW

***CBF constraint*:** $$L_f b_k(x(t)) + L_g b_k(x(t)) u(t) + \gamma(b_k(x(t))) \geq 0$$

Linear in u

This inequality provides a set of feedback controls that guarantee $b_k(x) \geq 0$ for all future times ⇨ FORWARD INVARIANCE

Details, Proofs, Applications in….



CONTROL BARRIER FUNCTIONS - OVERVIEW

- **Control Lyapunov Functions**:
  (can be used to stabilize one or more state variable)

CLF constraints: $L_f V(x(t)) + L_g V(x(t))u(t) + \varepsilon V(x(t))) \leq \delta(t)$

Example: $(v_i(t) - v_{ref}(t))^2$

Soft constraint parameter

CONTROL BARRIER FUNCTIONS - OVERVIEW

- **Solve optimization problem:**

$$\min_{\boldsymbol{u}(t),\delta(t)} \int_0^T ||\boldsymbol{u}(t)||^2 + p\delta^2(t)dt$$

subject to

$$\dot{x} = f(x) + g(x)u$$

State dynamics

$$L_f b_k(x(t)) + \ldots(t))u(t) + \gamma(b_k(x(t))) \geq 0$$

HARD safety constraints

CBF constraints

$$L_f V(x(t)) + L_g V(x(t))u(t) + \varepsilon V(x(t)) \ldots$$

SOFT constraints

CLF constraints

$$u_{min} \leq u \leq u_{max}$$

Control constraints

CONTROL BARRIER FUNCTIONS - OVERVIEW

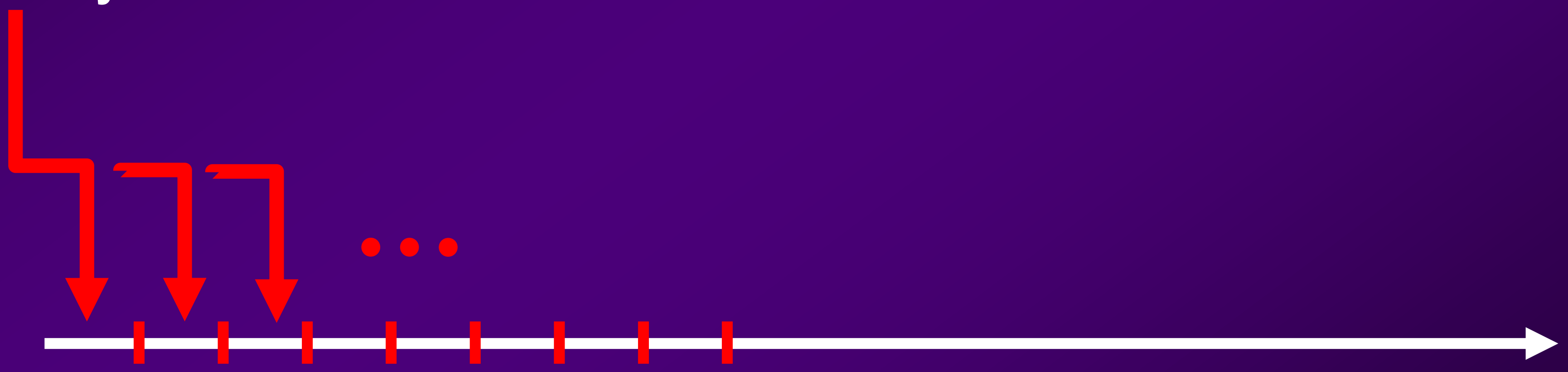
- **Discretize time and solve a sequence of Quadratic Programs (QPs):**

$$(\boldsymbol{u}^*(t_k), \delta^*(t_k)) = \arg \min_{\boldsymbol{u}(t_k), \delta(t_k)} \|\boldsymbol{u}(t_k)\|^2 + p\delta^2(t_k)$$

s.t. Vehicle Dynamics, CBF constraints, CLF constraints

Linear in u_k

- **Easy to solve**



SEVERAL ISSUES TO RESOLVE…

1. What if control $u(t)$ does not show up in CBF constraint?

$$L_f b_k(x(t)) + \underbrace{\boxed{L_g b_k(x(t))}}_{=0\ ???} u(t) + \gamma(b_k(x(t))) \geq 0 \qquad (*)$$

2. Is each QP problem always **FEASIBLE** ?

3. Since CBF constraint is only a sufficient condition for the original constraint, how **CONSERVATIVE** are the controls that come from $(*)$?

4. What if the environment/system is **TIME-VARYING**?

5. What if **DYNAMICS ARE UNKNOWN**?

ISSUE 1: HIGH ORDER CBFs (HOCBFs)

KEY IDEA: Keep taking derivatives of $b(x)$ until u shows up

Define a sequence of functions $\psi_i(x)$:

Class K function

$$\psi_0(\boldsymbol{x}) := b(\boldsymbol{x})$$

$$\psi_i(\boldsymbol{x}) := \dot{\psi}_{i-1}(\boldsymbol{x}) + \alpha_i(\psi_{i-1}(\boldsymbol{x})), \quad i \in \{1, \dots, m\}$$

m **: Relative degree of $b(x)$ = number of times it needs to be differentiated along its dynamics until the control u explicitly shows in the corresponding derivative.**

***HOCBF constraints*:**

$$\sup_{\boldsymbol{u} \in U} [L_f^m b(\boldsymbol{x}) + [L_g L_f^{m-1} b(\boldsymbol{x})]\boldsymbol{u} + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x}))] \geq 0$$

Lie derivatives along f and g

Remaining Lie derivatives

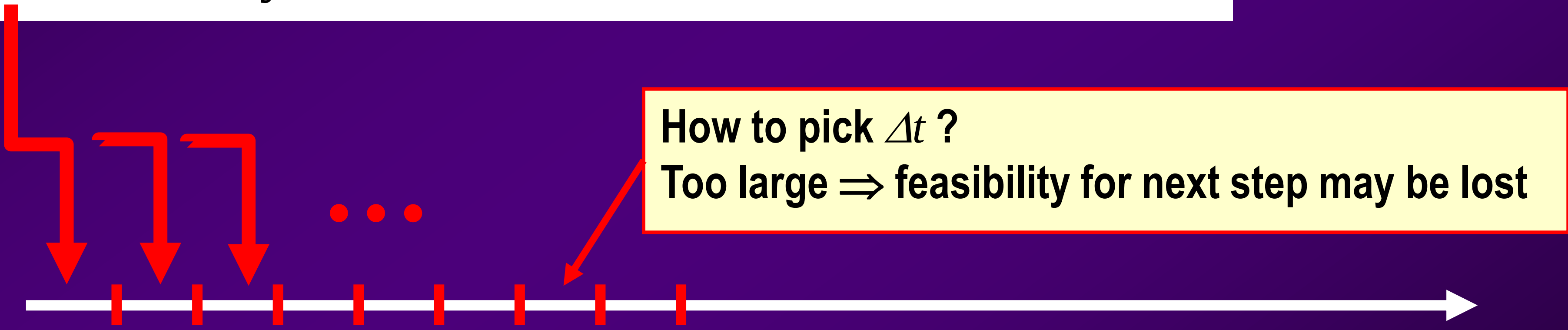
Class K function

ISSUE 2: QP FEASIBILITY → EVENT-DRIVEN CONTROL

KEY IDEA: REPLACE TIME-DRIVEN CONTROL BY EVENT-DRIVEN CONTROL

$$(\boldsymbol{u}^*(t_k), \delta^*(t_k)) = \arg \min_{\boldsymbol{u}(t_k),\delta(t_k)} \|\boldsymbol{u}(t_k)\|^2 + p\delta^2(t_k)$$

s.t. Vehicle Dynamics, CBF constraints, CLF constraints

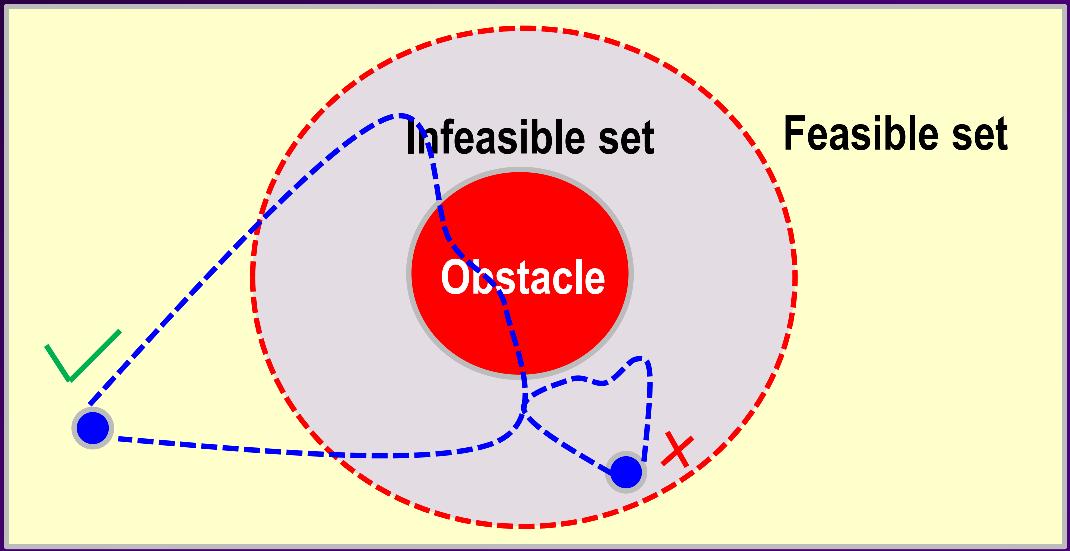


Instead, trigger the next QP based on appropriate EVENTS
Side benefit: increased unpredictability ⇒ extra security !

Xiao, Belta, Cassandras, *IEEE Trans. on Automatic Control*, 2023

ISSUE 3: CONSERVATIVENESS → ADAPTIVE CBFs

KEY IDEA: ADAPTIVE CBFs TO MAXIMIZE FEASIBILITY ROBUSTNESS



If constraint becomes active before detection of obstacle, then safety is not guaranteed *(e.g., too late to decelerate/turn away)*

Can we shrink infeasible set? (which may be hard to identify…)
$\Rightarrow$ Make constraint active **as late as possible**

Define a FEASIBILITY ROBUSTNESS metric and adapt CBF constraint to maximize it

ADDED CHALLENGE: *Learn INFEASIBLE SETS*

ISSUE 4: TIME-VARYING ENVIRONMENTS → ADAPTIVE CBFs

KEY IDEA: ADAPTIVE CBFs

Revisit ADAPTIVE CBFs with more flexible forms of adaptivity, e.g.,

$$L_f^m b_j(\boldsymbol{x}) + L_g L_f^{m-1} b_j(\boldsymbol{x})\boldsymbol{u} + O(b_j(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x})) \geq 0$$

$$\psi_i(\boldsymbol{x}) := \dot{\psi}_{i-1}(\boldsymbol{x}) + p_i \psi_{i-1}^{q_i}(\boldsymbol{x}), \quad i \in \{1, \dots, m\}$$

Non-negative parameters

Learn optimal parameters

Xiao, Belta, Cassandras, *Automatica*, 2022

ISSUE 5: UNKNOWN DYNAMICS → EVENT-DRIVEN CONTROL

1. Define **ADAPTIVE AFFINE DYNAMICS** $\dot{\bar{\boldsymbol{x}}} = f_a(\bar{\boldsymbol{x}}) + g_a(\bar{\boldsymbol{x}})\boldsymbol{u}$

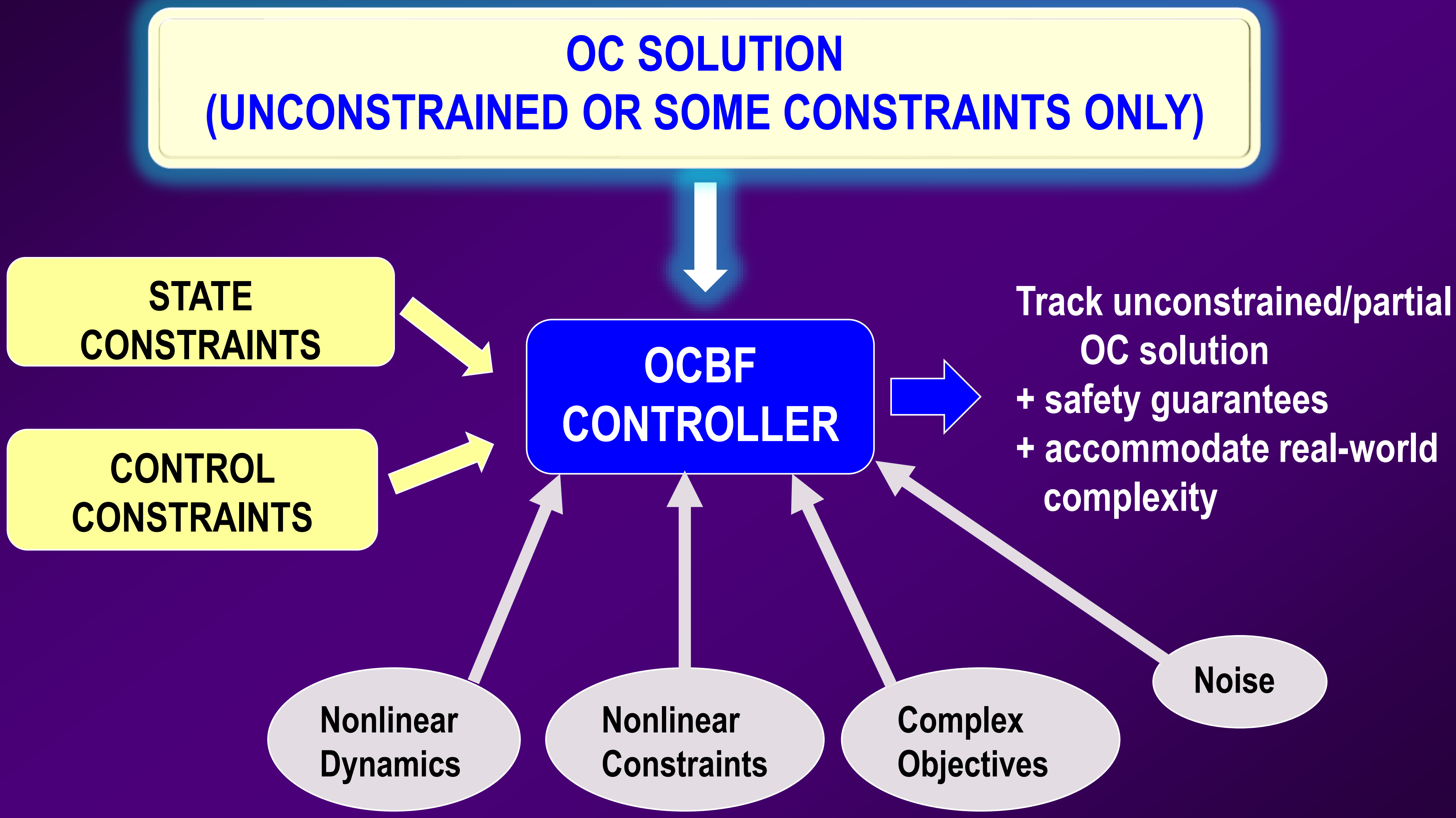
2. Find HOCBF that guarantees $b(\boldsymbol{x}(t)) \geq 0, \forall t \in [0, T]$

3. Formulate CBF-based QP

4. Determine EVENTS required to solve QPs + condition that guarantees $b(\boldsymbol{x}(t)) \geq 0, \forall t \in [0, T]$ between events

Xiao, Belta, Cassandras, *IEEE Trans. on Automatic Control*, 2023

OC + CBF APPROACH (OCBF): OVERVIEW



OPTIMAL CONTROL

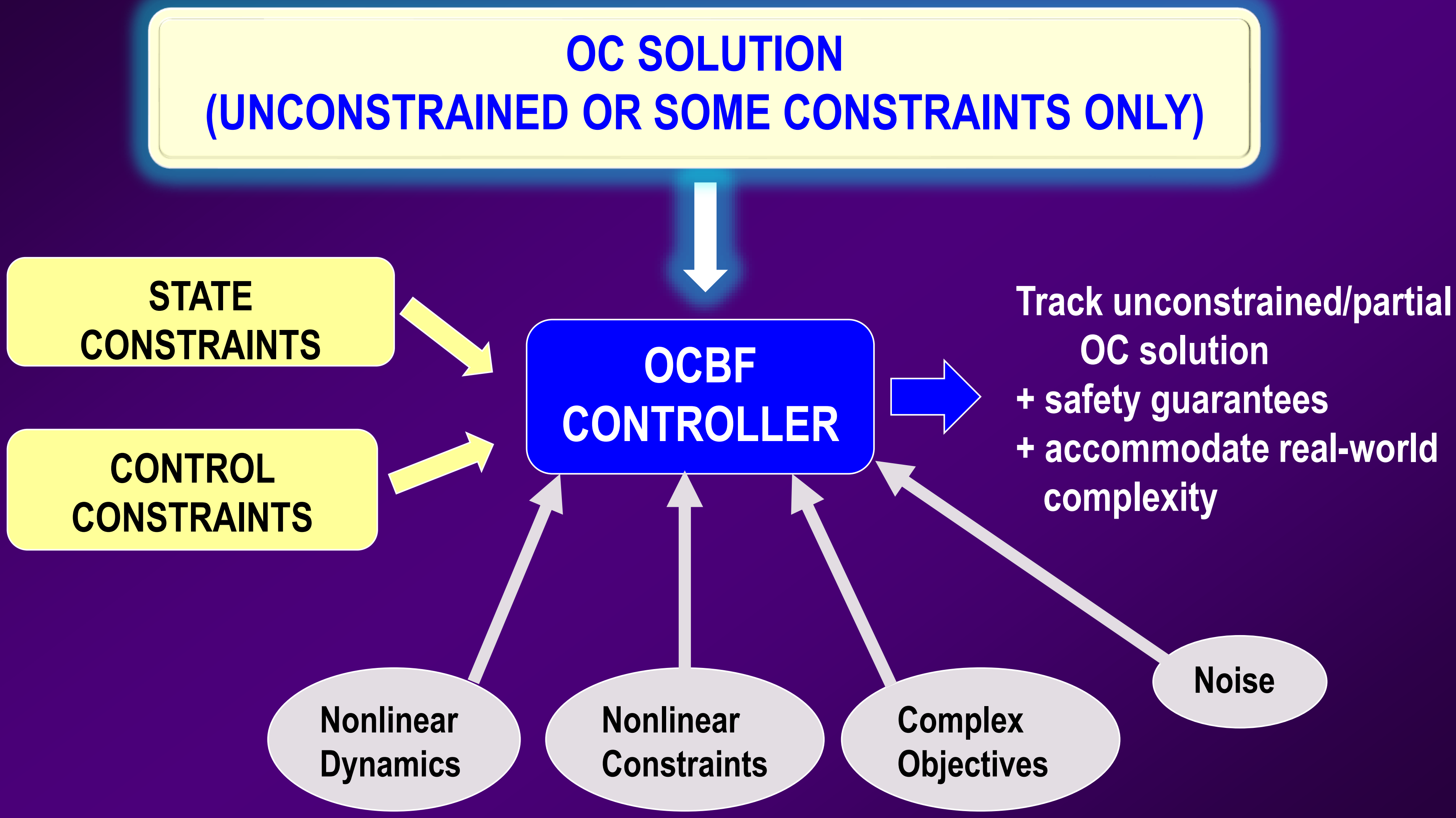
+

CBFs

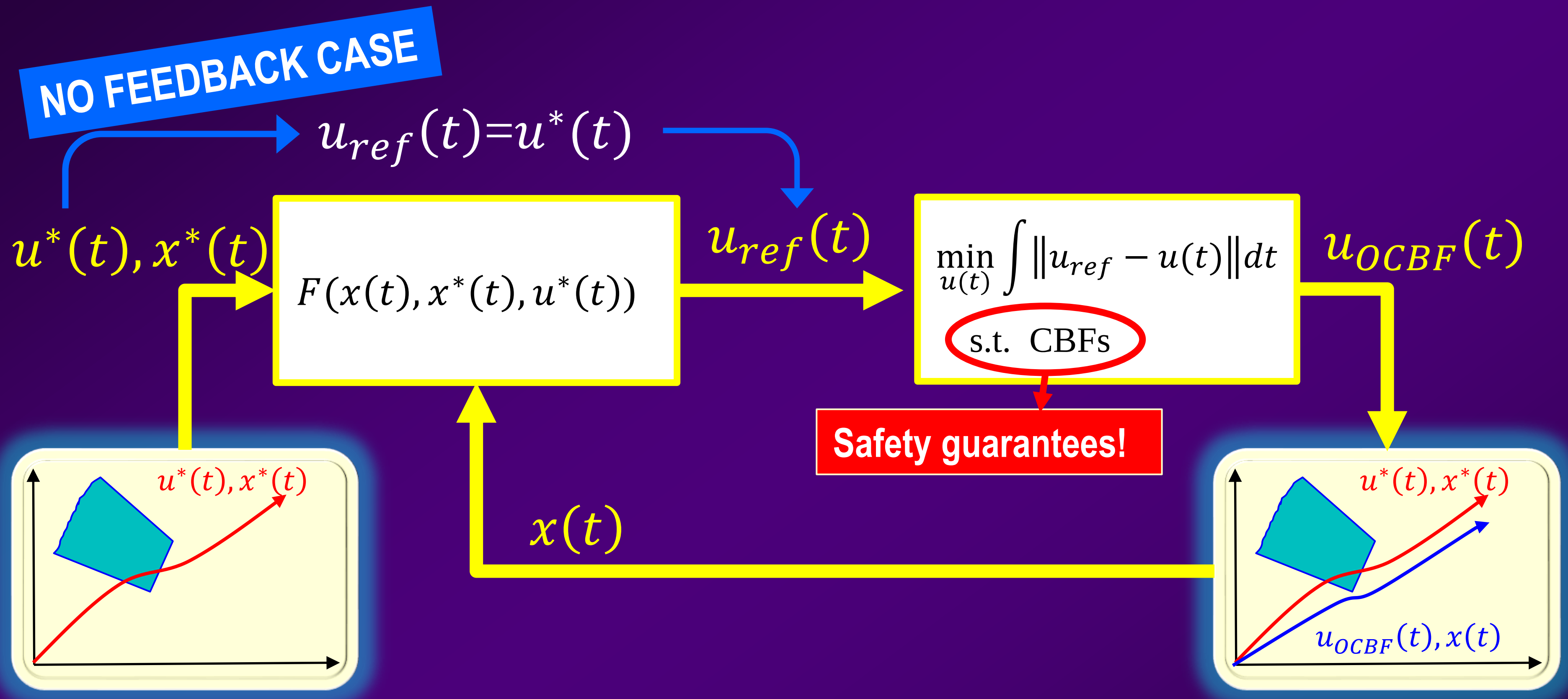
⬇

OCBF CONTROL

OC + CBF APPROACH (OCBF): OVERVIEW



OCBF CONTROL SYSTEM



GET BEST POSSIBLE OCP SOLUTION WITHIN REAL-TIME CONSTRAINTS → GET FEEDBACK → OPT. TRACK WITH GUARANTEED SAFETY

OC + CBF APPROACH (OCBF)

STEP 1: Solve the Optimal Control Problem (OCP)

$$J = \int_{t_0}^{t_f} [\beta + C(\boldsymbol{x}, \boldsymbol{u}, t)]\, dt,$$

subject to:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}$$

$$\boldsymbol{x}(t_f) = \bar{X}, \quad \text{(or min. } \|\boldsymbol{x}(t_f) - \bar{X}\|^2 \text{)}$$

$$b_j(\boldsymbol{x}(t)) \geq 0, \quad \forall t \in [t_0, t_f]$$

$$\boldsymbol{x}_{\min} \leq \boldsymbol{x}(t) \leq \boldsymbol{x}_{\max}, \forall t \in [t_0, t_f]$$

$$\boldsymbol{u}_{min} \leq \boldsymbol{u} \leq \boldsymbol{u}_{max}$$

…with the least possible amount of simplifications, *e.g., linear dynamics and no constraints*.

$$\boxed{\boldsymbol{u}^*(t) \text{ and } \boldsymbol{x}^*(t),}$$

OC + CBF APPROACH (OCBF)

STEP 2: Observe actual system, allowing for unmodeled disturbances and noise

$$\dot{x} = f(x) + g(x)u + w$$

and define $u_{ref}(t) = h(u^*(t), x^*(t), x(t))$

From OCP

Actual state trajectory

EXAMPLES:

$$u_{ref}(t) = \sum_{j \in \{1,\ldots,n\}} \frac{x_j^*(t)}{x_j(t)} u^*(t)$$

$$u_{ref}(t) = e^{\sum_{j=1}^{n} \frac{x_j^*(t) - x_j(t)}{\sigma_j}} u^*(t)$$

…possibly simply $u_{ref}(t) = u^*(t)$

OC + CBF APPROACH (OCBF)

STEP 3: Optimally track the simplified OCP solution using actual state feedback for improved performance

$$\min_{u_k(t),\delta_k(t)} \int_{t_0}^{t_f} \left(\beta\delta_k^2(t) + \|\boldsymbol{u}(t) - \boldsymbol{u}_{ref}(t)\|^2\right) dt.$$

subject to:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{w} \qquad \boldsymbol{u}_{min} \leq \boldsymbol{u} \leq \boldsymbol{u}_{max}$$

$$L_f^{m_j} b_j(\boldsymbol{x}) + L_g L_f^{m_j-1} b_j(\boldsymbol{x})\boldsymbol{u} + S(b_j(\boldsymbol{x})) + p_{i,safe}\alpha_{m_j}(\psi_{m_j-1}(\boldsymbol{x})) \geq 0,\ j \in S_o$$

$$L_f^{m_i} b_{i,\max}(\boldsymbol{x}) + L_g L_f^{m_i-1} b_{i,\max}(\boldsymbol{x})\boldsymbol{u} + S(b_{i,\max}(\boldsymbol{x})) + p_{i,\max}\alpha_{m_i}(\psi_{m_i-1}(\boldsymbol{x})) \geq 0$$

$$L_f^{m_i} b_{i,\min}(\boldsymbol{x}) + L_g L_f^{m_i-1} b_{i,\min}(\boldsymbol{x})\boldsymbol{u} + S(b_{i,\min}(\boldsymbol{x})) + p_{i,\min}\alpha_{m_i}(\psi_{m_i-1}(\boldsymbol{x})) \geq 0$$

$$L_f V(y_k(t)) + L_g V(y_k(t))\boldsymbol{u}(t) + \epsilon V(y_k(t)) \leq \delta_k(t)$$

CLF for some states (if desired)
e.g., $y_i(t) := v_i(t) - v_{max}$

HOCBFs corresponding to state and safety constraints: $b_j(\boldsymbol{x}(t)) \geq 0,\ \boldsymbol{x}_{\min} \leq \boldsymbol{x}(t) \leq \boldsymbol{x}_{\max}$

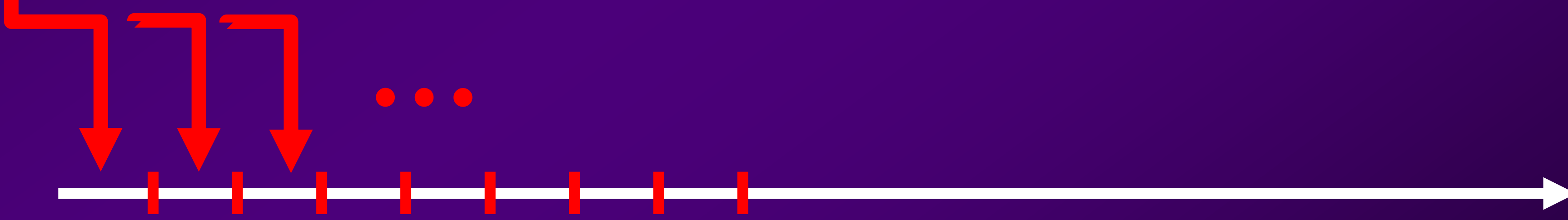
OC + CBF APPROACH (OCBF)

STEP 4: **Solve the optimal tracking problem in real time**

Discretize time and solve a sequence of Quadratic Programs (QPs):

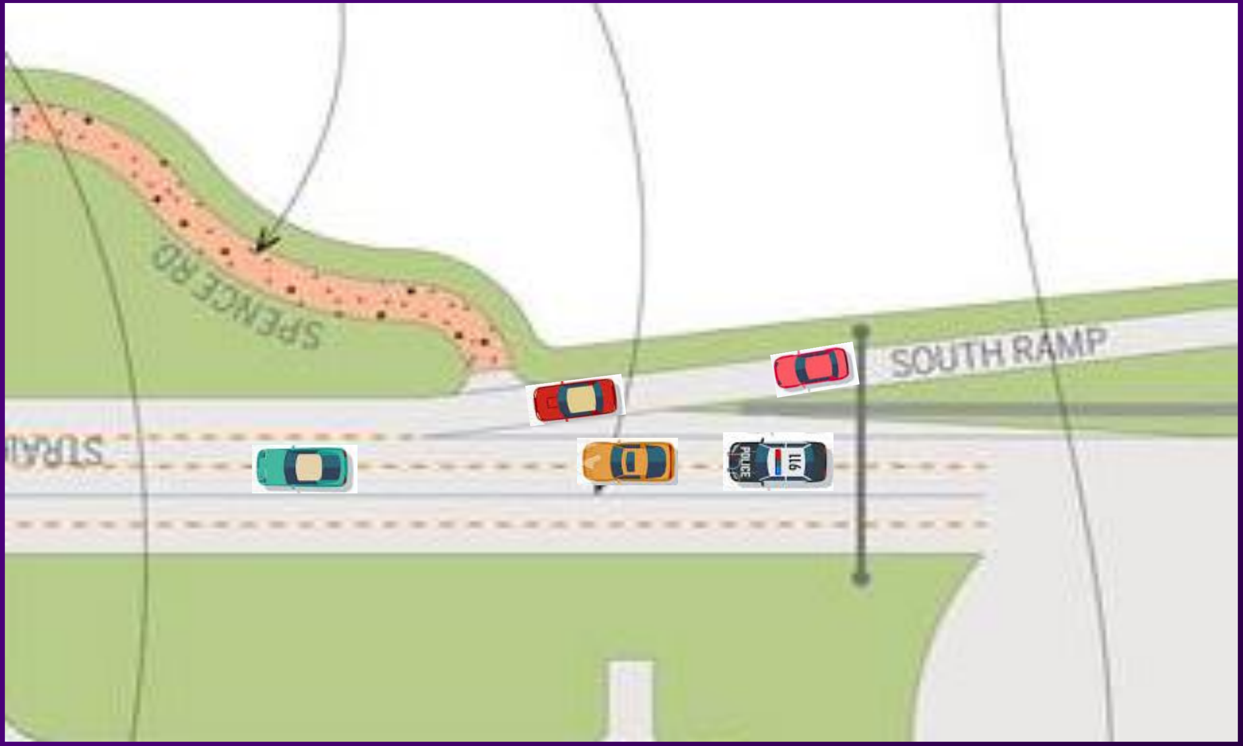
$$(\boldsymbol{u}^*(t_k), \delta^*(t_k)) = \arg \min_{\boldsymbol{u}(t_k), \delta(t_k)} \|\boldsymbol{u}(t_k)\|^2 + p\delta^2(t_k)$$

$$\text{s.t.} \quad \text{Vehicle Dynamics, CBF constraints, CLF constraints}$$

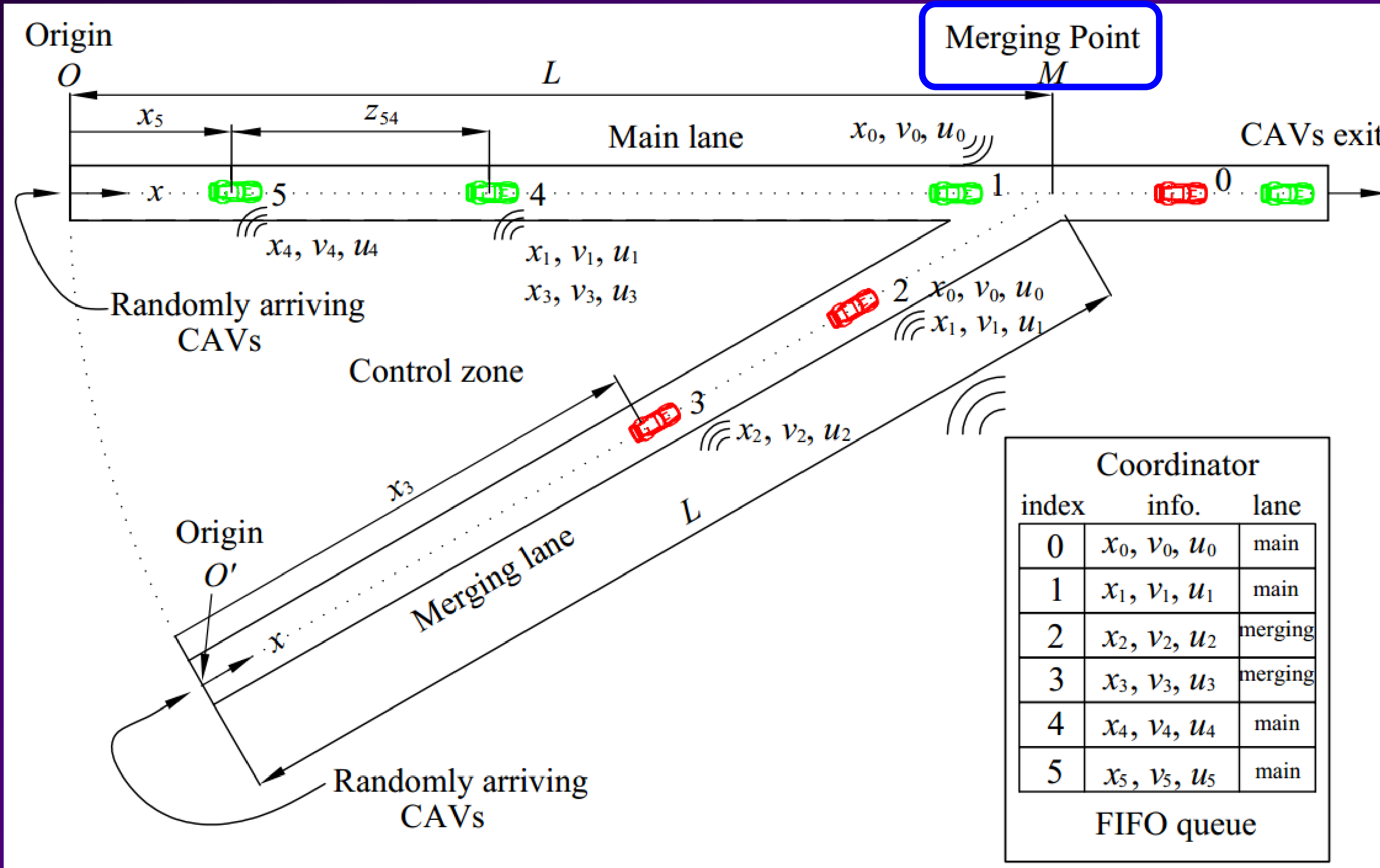


Preferably, using EVENT-DRIVEN control

MERGING AS AN OPTIMAL CONTROL PROBLEM



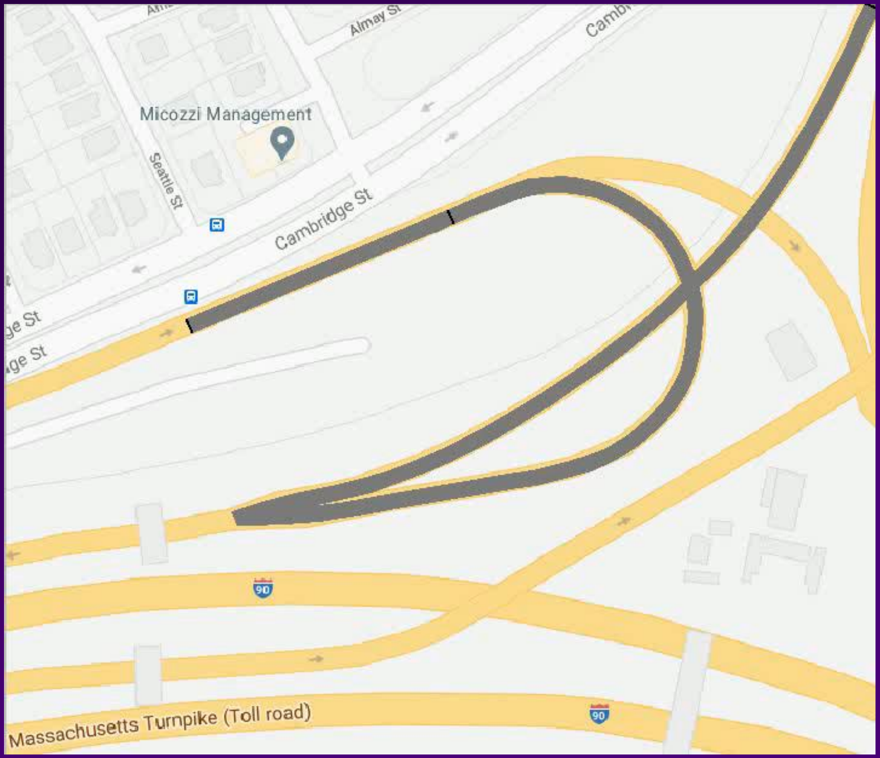
OPTIMAL MERGING

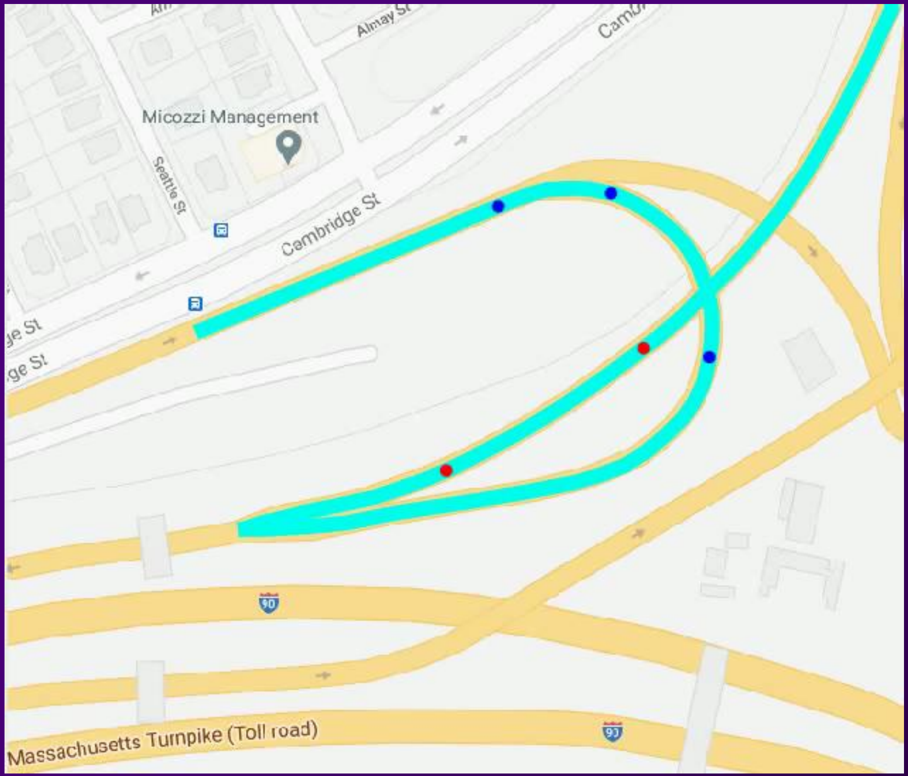


Xiao and Cassandras, *Automatica 2021*

MERGING WITH CURVES

Merging into the Mass Turnpike in Boston, MA



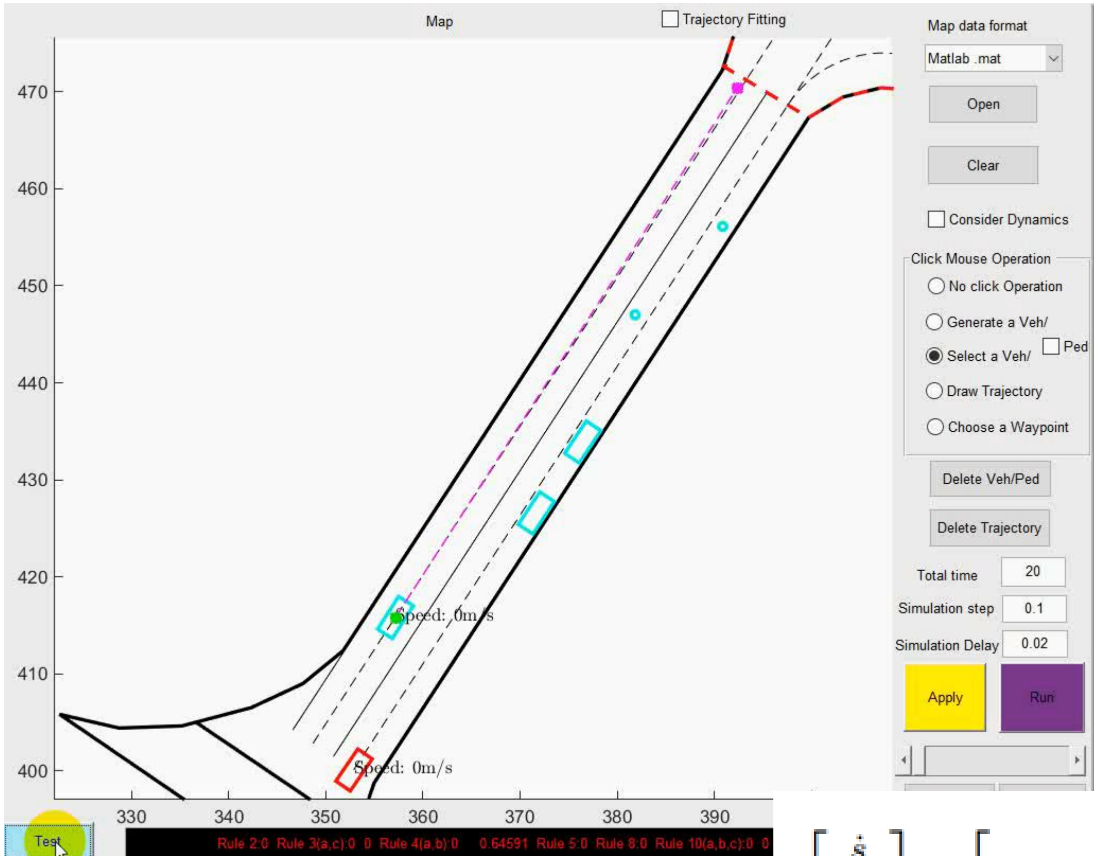


- **Human-driven vehicles**

- **CAVs**

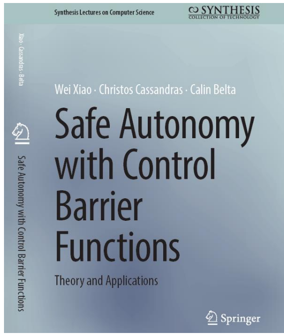
  (with comfort accounted for to accommodate centrifugal forces)

Xiao and Cassandras, *CDC, 2021*



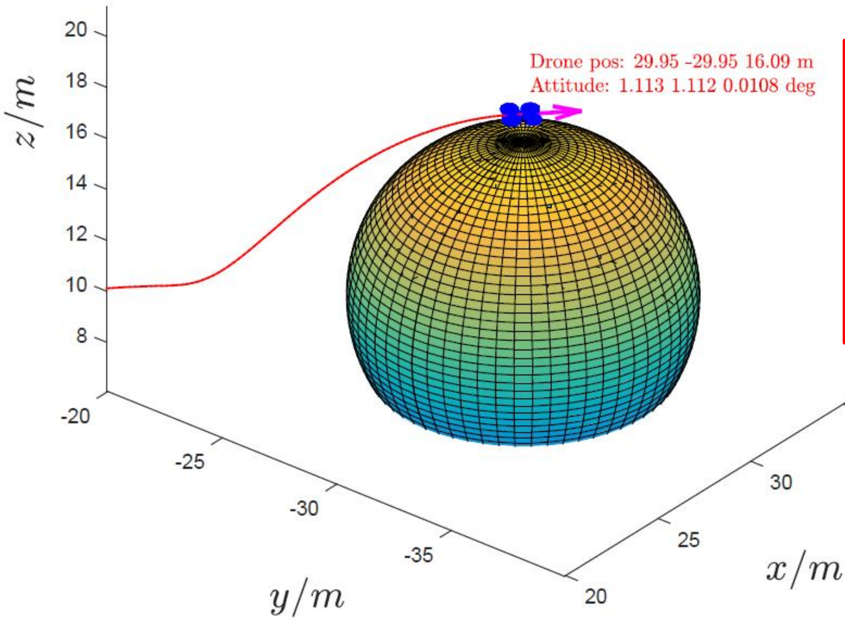
- **Trajectory tracking (mid-lane)**
- **State constraints (steering angle, speed)**
- **Control bounds (jerk, steering accel.)**
- **Obey prioritized list of traffic rules**

Details:
Ch. 10 of …



$$\underbrace{\begin{bmatrix}\dot{s}\\ \dot{n}\\ \dot{\mu}\\ \dot{v}\\ \dot{a}\\ \dot{\delta}\\ \dot{\omega}\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}\frac{v\cos(\mu+\beta)}{1-n\kappa}\\ v\sin(\mu+\beta)\\ \frac{v}{l_r}\sin\beta-\kappa\frac{v\cos(\mu+\beta)}{1-n\kappa}\\ a\\ 0\\ \omega\\ 0\end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}}_{g(x)} \underbrace{\begin{bmatrix}u_{jerk}\\ u_{steer}\end{bmatrix}}_{u}$$

QUADROTOR SAFE NAVIGATION



$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \\ \dot{y} \\ \dot{v}_y \\ \dot{z} \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} v_x \\ 0 \\ v_y \\ 0 \\ v_z \\ -g \end{bmatrix} + \frac{F}{M} \begin{bmatrix} 0 \\ C_\psi S_\theta C_\phi + S_\psi S_\phi \\ 0 \\ S_\psi S_\theta C_\phi - C_\psi S_\phi \\ 0 \\ C_\theta C_\phi \end{bmatrix} - \frac{1}{M} \begin{bmatrix} 0 & 0 & 0 \\ A_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_x \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

16-state quadrotor model

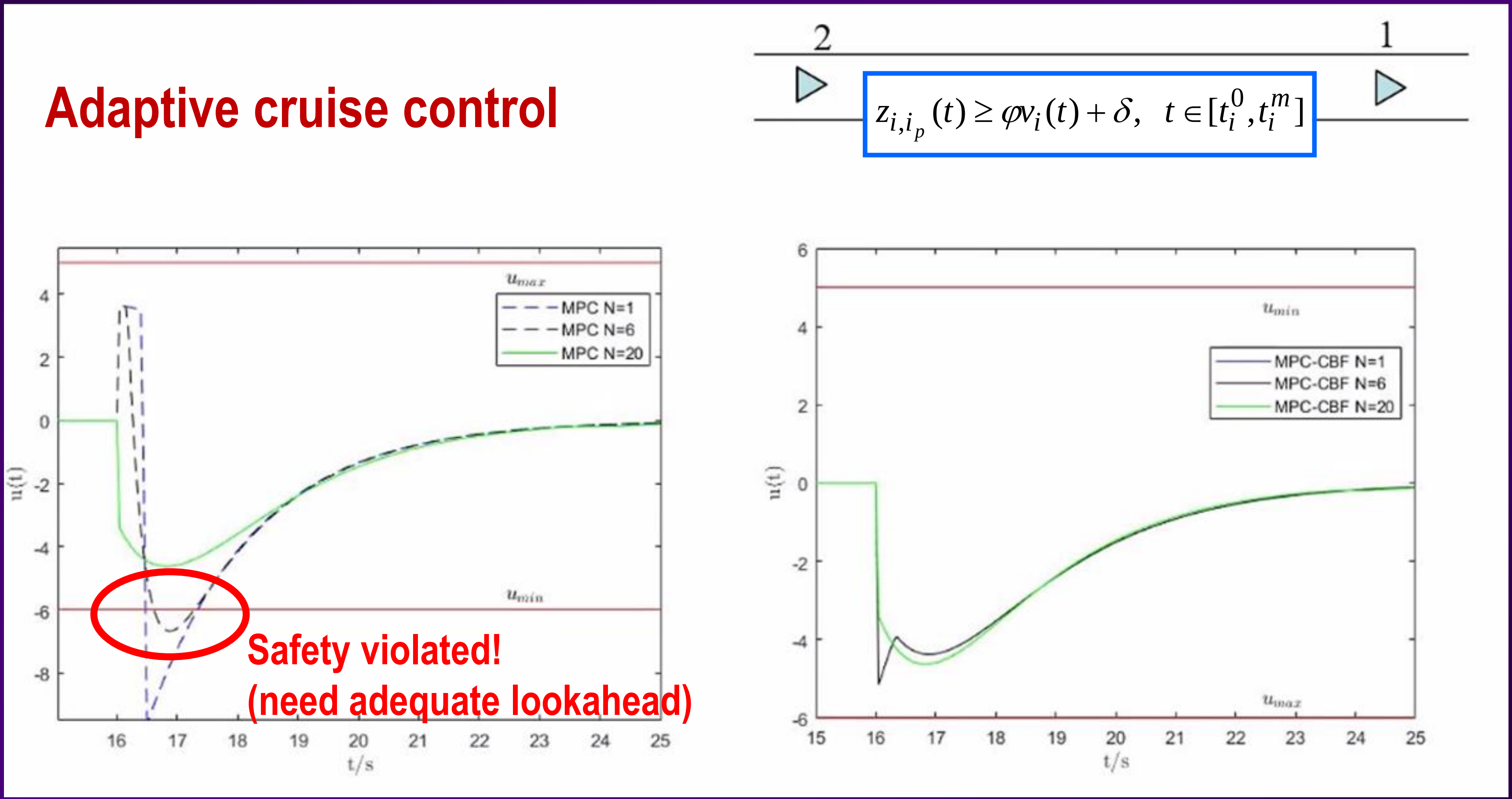
Roll, Pitch, Yaw

$$\begin{bmatrix} \dot{\phi} \\ \dot{\omega}_\phi \\ \dot{\theta} \\ \dot{\omega}_\theta \\ \dot{\psi} \\ \dot{\omega}_\psi \end{bmatrix} = \begin{bmatrix} \omega_\phi & 0 & 0 \\ 0 & \omega_\phi C_\phi T_\theta + \omega_\theta S_\phi / C_\theta^2 & -\omega_\phi S_\phi C_\theta + \omega_\theta C_\phi / C_\theta^2 \\ \omega_\theta & 0 & 0 \\ 0 & -\omega_\phi S_\phi & -\omega_\phi C_\phi \\ \omega_\psi & 0 & 0 \\ 0 & \omega_\phi C_\phi / C_\theta + \omega_\theta S_\phi T_\theta / C_\theta & -\omega_\phi S_\phi / C_\theta + \omega_\theta C_\phi T_\theta / C_\theta \end{bmatrix} \begin{bmatrix} 1 \\ q \\ r \end{bmatrix} + W_\eta^{-1} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}$$

Angular speed of each rotor

$$\dot{\omega}_i = b_1 u_i - \beta_0 - \beta_1 \omega_i - \beta_2 \omega_i^2, i \in \{1, 2, 3, 4\}$$

- Make CBF-based methods less myopic, less conservative: MPC + CBFs



Computational cost: Generally, MPC cost > 10 × (CBF-based cost)

THE ROAD AHEAD…

- Resolve all practical issues related to CBFs: conservativeness, feasibility, unknown dynamics, etc
- Using learning methods: Safety guarantees for NNs
- Reduce complexity: Safety Attention Transformations
  - Pay more "attention" to some state variables
  - Replace HOCBFs (can be complex) by CBFs